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Deformations in Closed String Theory — Canonical Formulation and Regularization

Martin Cederwall, Alexander von Gussich and Per Sundell

Institute for Theoretical Physics
Chalmers University of Technology and Göteborg University
S-412 96 Göteborg, Sweden

Abstract: We study deformations of closed string theory by primary fields of conformal weight $(1,1)$, using conformal techniques on the complex plane. A canonical surface integral formalism for computing commutators in a non-holomorphic theory is constructed, and explicit formulæ for deformations of operators are given. We identify the unique regularization of the arising divergences that respects conformal invariance, and consider the corresponding parallel transport. The associated connection is metric compatible and carries no curvature.

email addresses: tfemc@fy.chalmers.se
tfeavg@fy.chalmers.se
tfepsu@fy.chalmers.se

1. Introduction and Summary

Probably the most important theoretical problem concerning string theory is the lack of a "covariant" formulation. Despite the fact that closed string theory contains gravity as part of the infinite spectrum, there is no formulation of string theory that is manifestly invariant under general coordinate transformations. It is likely that a fundamental gauge principle of closed string theory involves some quantum geometric invariance generalizing that of Einstein's theory of gravity. The most promising place to look for such gauge symmetries is closed string field theory [1]. However, despite the interesting algebraic structures arising in the field theory formulation of string theory, some aspects are still simpler in a first-quantized version. A scattering amplitude that in the first-quantized framework is given by a single integral with vertex operator insertions decomposes into several terms in the field theory, due to the somewhat arbitrary decomposition of a Riemann surface into propagator and vertex parts. The present work is performed entirely in a first-quantized formalism, but it is possible to translate it into a field theoretic language. We will address that issue in a forthcoming paper [2], and only comment on the connection in this paper.

The purpose of the paper is to investigate the local structure of "string theory space", i.e., the space of consistent backgrounds for string propagation. To this end, we consider "deformations" of closed string theory, where the flat background is shifted to some infinitesimal field configuration corresponding to physical states in the string theory, in which a free string propagates.

It is a priori difficult to judge what gauge transformations should look like, simply because we do not know what the ultimate off-shell field content is. However, the situation may be better than expected. We will demonstrate that deformations of closed string theory, i.e., transformations between inequivalent backgrounds, are "almost inner automorphisms" of the conformal field theory — they are generated by the action of an operator formed from the oscillators in the theory itself in a given background, and behave analytically on momentum operators. Each such operator has a regular action on almost all operators in the theory — there are only simple poles for certain momentum eigenvalues (resonances).

In order to construct these operators and investigate their action, we determine how the apparent divergences arising at various stages in the calculations should be regularized. The guideline is conformal invariance, and the answer is unique: analytic continuation. Only logarithmic divergences survive and produce simple poles, all other divergences are uniquely regularized to give finite results. This is an important issue to settle; there has been questions both about the freedom to choose regularization and what it should look like [3,4,5,6]. In the previous formulations the simple poles are regarded as nonregular terms which need to be subtracted. This introduces an arbitrariness in the choice of finite counterterms constrained by conformal invariance, though some preferred connections have been found. The subtractions are not necessery if L_0 has a continous spectrum. Then the produced simple poles have a well defined meaning as distributions in the external momentum decomposition of the background perturbation. This is a physical assumption which is analogous to the treatment of the delta functions in the scattering matrix using adiabatic turn on of the perturbation. In this picture the perturbation is localized in space-time such that the incoming and outgoing states are unperturbed. Regularization by analytic continuation has also been considered in four dimensions, and has been shown to be equivalent to dimensional regularization [7].

Since we will use a canonical framework, we will frequently be calculating commutators of operators in the non-holomorphic conformal field theory. To streamline these calculations, we develop a formalism that translate operator products into commutators, analogously to what is

done in a holomorphic theory with contour integrals (our generators are naturally surface integrals of local operators). This makes it possible to take advantage of the simple form of correlation functions for radial quantization on the complex plane. This property is lost when one goes to equal-time quantization in cylindrical coordinates, and one is naturally lead to calculations involving distributions.

Deformations of closed string theory with physical vertex operators as automorphisms of the double Virasoro algebra were first considered in [8]. In cylindrical coordinates they take the form

$$\delta_\Phi T(\sigma) = \delta_\Phi \bar{T}(\sigma) = \Phi(\sigma) \ , \tag{1.1}$$

where Φ is a physical vertex operator, a primary field of weight $(1,1)$. If one goes to planar variables, one obtains

$$\begin{aligned} (\delta_\Phi T)(z,\bar{z}) &= \frac{\bar{z}}{z}\Phi(z,\bar{z}) \ , \\ (\delta_\Phi \bar{T})(z,\bar{z}) &= \frac{z}{\bar{z}}\Phi(z,\bar{z}) \ , \end{aligned} \tag{1.2}$$

where the factors in front of Φ arise because of the differences in conformal weights between Φ and T or $\bar{T}$, respectively. The most "covariant" formulation has been given in [9], where deformations of surface states of given genus and number of punctures are expressed in terms of surface states of the same genus and one more puncture. The $(1,1)$-form $\phi = \Phi(z,\bar{z})dz\wedge d\bar{z}$ is inserted at the extra puncture and its position is integrated over. This formulation is the one that is most suited for string field theory. Then the N-string vertex at genus g gets a modification that comes from the $(N+1)$-string vertex at genus g. The transformations act naturally on the second-quantized (multi-string) Fock space, and the problem with potential divergences from colliding punctures is pushed ahead. In this paper we work in a first-quantized framework, where the divergences are taken care of immediately, without cutting out semi-infinite propagators from the amplitudes. The transformations act on the first-quantized (one-string) Hilbert space, and the deformed theory is seen as a single string moving in a non-trivial background. However, we find relations for amplitudes that, thanks to the regularization we use, provide a natural link to a second-quantized formalism. We will adress this in more detail in the forthcoming paper [2].

When we consider repeated tranformations, we find that the commutator of two deformations vanishes identically on the full state space, i.e. if the deformation is regarded as a parallell transport along directions in the space of conformal field theories then the curvature vanishes.

We think that the simple technique we have developed for calculating commutators in a non-holomorphic theory has a potential to solve problems associated with closed string theory, especially those connected to non-holomorphic, "bilateral" operators [10,11]. We intend to use it in the search for generalizations of the general coordinate invariance [12], possibly involving operators at all mass levels. It should also be suited for posing questions about finite deformations (finite parallell transport) and more general deformations into non-conformal theories.

2. Canonical Formulation

Under the deformation by the primary (1,1)-field $\Phi(z,\bar{z})$ the holomorphic and anti-holomorphic components of the stress tensor transform as

$$\begin{aligned} T(z) &\longrightarrow T(z) + \varepsilon\frac{\bar{z}}{z}\Phi(z,\bar{z}) \ , \\ \bar{T}(\bar{z}) &\longrightarrow \bar{T}(\bar{z}) + \varepsilon\frac{z}{\bar{z}}\Phi(z,\bar{z}) \ . \end{aligned} \tag{2.1}$$

It is of course interesting to see if these transformations can be seen as inner derivations (infinitesimal inner automorphisms), i.e., if they are generated by the adjoint action of some generator ϱ_Φ constructed from the fields in the theory.

Consider a general physical vertex operator $\Phi(z,\bar{z})$, carrying (left and right) momentum k. It can be given a mode expansion as

$$\Phi(z,\bar{z}) = \sum_{m,n\in\mathbb{Z}} \Phi_{mn}|z|^{2\gamma}z^{-1-m}\bar{z}^{-1-n} \ , \tag{2.2}$$

where γ is the operator valued shift $(k\cdot p)/4$. The stress tensor has the expansions

$$\begin{aligned} T(z) &= \sum_{m\in\mathbb{Z}} L_m z^{-2-m} \ , \\ \bar{T}(z) &= \sum_{m\in\mathbb{Z}} \bar{L}_m \bar{z}^{-2-m} \ . \end{aligned} \tag{2.3}$$

Since the physical vertex operators factorize in holomorphic and anti-holomorphic parts, we actually have $\Phi_{mn} = V_m\bar{V}_n$, but that expression will not be needed here. The commutators of the modes of T or $\bar{T}$ and Φ are

$$\begin{aligned} [L_m,\Phi_{nl}] &= (\gamma-n)\Phi_{n+m,l} \ , \\ [\bar{L}_m,\Phi_{nl}] &= (\gamma-l)\Phi_{n,l+m} \ . \end{aligned} \tag{2.4}$$

One may try to construct from the modes of Φ an operator whose adjoint action gives the variations (2.1) in the stress tensor. Upon doing this, one must remember that commutators have to be evaluated at "equal time", here meaning equal radius. We denote this radius by R, and the generator of deformations corresponding to the field Φ at radius R by $\varrho_\Phi(R)$. We also let $\delta_\Phi(R) = \mathrm{ad}\varrho_\Phi(R)$.

Expansion of the transformations (2.1) on the circle $|z| = R$ reads

$$\begin{aligned} \delta_\Phi(R)L_m &= \oint_{|z|=R} \frac{dz}{2\pi i} z^{1+m}\frac{\bar{z}}{z}\Phi(z,\bar{z}) = \sum_{n\in\mathbb{Z}} R^{2(\gamma-n)}\Phi_{n+m,n} \ , \\ \delta_\Phi(R)\bar{L}_m &= \oint_{|z|=R} \frac{d\bar{z}}{2\pi i} \bar{z}^{1+m}\frac{z}{\bar{z}}\Phi(z,\bar{z}) = \sum_{n\in\mathbb{Z}} R^{2(\gamma-n)}\Phi_{n,n+m} \ . \end{aligned} \tag{2.5}$$

It is worth stressing that the deformed components of the stress tensor do not respect any holomorphicity conditions. The actual forms of the deformed L_m and $\bar{L}_m$ depend on the radius R. This is natural — their unitary time evolutions are governed by the deformed time-dependent hamiltonian $L_0 + \bar{L}_0 + \delta_\Phi(R)(L_0 + \bar{L}_0)$. It is easily verified that the new L_m's and $\bar{L}_m$'s satisfy $Vir \oplus Vir$, i.e., $\delta_\Phi(R)$ is a derivation of the double Virasoro algebra.

Using (2.4), we notice that the variations (2.1) are formally generated by

$$\varrho_\Phi(R) = -\sum_{n\in\mathbb{Z}} \frac{R^{2(\gamma-n)}}{\gamma-n}\Phi_{nn} \ . \tag{2.6}$$

The denominators in (2.6) tell us that $\varrho_\Phi(R)$ has operator valued poles whenever $\gamma\in\mathbb{Z}$, that is when $\varrho_\Phi(R)$ acts on a state whose momentum k' satisfies $(k\cdot k')/4\in\mathbb{Z}$. This means that although the adjoint action $\delta_\Phi(R) = \mathrm{ad}\varrho_\Phi(R)$ is a derivation of $Vir \oplus Vir$, it is not, strictly speaking, an inner derivation of the entire conformal field theory — the presence of poles is what makes the

transformations non-trivial. We will make repeated use of the analytic dependence on the mode shift γ.

We already know from [9], as described in the introduction, that deformations are related to integrals of the $(1,1)$-form $\phi = \Phi(z,\bar{z})dz\wedge d\bar{z}$. Equation (2.6) can in fact be written

$$\varrho_\Phi(R) = -\frac{1}{\pi}\int\limits_{|z|<R} d^2z\,\Phi(z,\bar{z}) = \frac{1}{2\pi i}\int\limits_{|z|<R}\phi\ . \qquad (2.7)$$

To establish equality of equations (2.6) and (2.7) we are led to introduce the fundamental regularization used in this paper.

3. Regularization

The regularization we will use is defined through analytic continuation in the mode shift γ. We introduce it by verifying eq. (2.7). Explicit calculation of the right hand side yields

$$\begin{aligned}
-\frac{1}{\pi}\int\limits_{|z|<R} d^2z\,\Phi(z,\bar{z}) &= -\frac{1}{\pi}\int\limits_{|z|<R} d^2z \sum_{m,n\in\mathbb{Z}}\Phi_{mn}|z|^{2\gamma}z^{-1-m}\bar{z}^{-1-n}\\
&= -\frac{1}{\pi}\int\limits_0^R rdr\int\limits_0^{2\pi}d\theta\sum_{m,n\in\mathbb{Z}}\Phi_{mn}r^{2\gamma-2-m-n}e^{i(n-m)\theta}\\
&= -\sum_{n\in\mathbb{Z}}\frac{R^{2(\gamma-n)}}{\gamma-n}\Phi_{nn}\ .
\end{aligned} \qquad (3.1)$$

The prescription for the radial integration is

$$\int\limits_0^1 dx\,x^\alpha = \frac{1}{1+\alpha}\ ,\quad \alpha\neq -1\ , \qquad (3.2)$$

and it is obtained through analytic continuation from the true region of convergence, $\alpha > -1$. In terms of the primitive functions, it corresponds to setting $x^{\alpha+1}|_{x=0} = 0$, while $\log x|_{x=0}$ is undetermined. The only remaining divergences that can not (and should not) be regularized this way are the logarithmic ones responsible for the pole in (3.2). The prescription for evaluating surface integrals is to first perform the angular integration (to eliminate potential poles with zero residue) and then regularize the radial integration according to (3.2).

Some comments are in order.

This type of regularization is exactly the one used for calculation of amplitudes in string theory. When calculating e.g. a four-string amplitude by integrating over the position of one of the vertex operators (this type of integral will be discussed below), one encounters divergences when it approaches the locus of one of the others. The actual convergence region of the integral is a bounded region for the Mandelstam variables, that does not contain any resonances. Not until the result is analytically continued in momenta does the pole structure, exhibiting resonances in the different channels, arise. This analytic continuation amounts exactly to (3.2).

Another point worth mentioning is that as a corollary of (3.2) one has

$$\int_0^\infty dx\, x^\alpha = \left(\int_0^1 + \int_1^\infty \right) dx\, x^\alpha = \int_0^1 dx\, (x^\alpha + x^{-2-\alpha}) = \frac{1}{1+\alpha} + \frac{1}{-1-\alpha} = 0\ . \tag{3.3}$$

We should stress here that although the calculation is not valid for $\alpha = -1$ (there is a delta function interpretation), the formalism allows us to care about the integral only as an analytic function of α. Then the singularity at $\alpha = -1$ is removable, and eq. (3.3) is valid for all α through analytic continuation. For a field $\Phi(z,\bar{z})$ on the the complex plane with singularities only at $z = 0$ and ∞, the statement (3.3) translates into

$$\int_{\mathbb{C}} d^2 z\, \Phi(z,\bar{z}) = 0\ . \tag{3.4}$$

If one is not used to calculating string amplitudes, the prescription (3.2) may look far-fetched. One may then consider its meaning in cylindrical coordinates, where time is Wick-rotated back to Minkowski signature. Then eq. (3.3) amounts to $\int_{-\infty}^{\infty} dt\, e^{i\beta t} = 0$ as an analytic function of β, which is a less surprising statement.

Furthermore, this regularization is closely related to ζ-function regularization, that is commonplace in string theory, in that both just define analytic continuations of sums or integrals away from the regions of convergence. We will comment on this connection later.

Finally, the analytic behaviour of $\varrho_\Phi(R)$ in γ makes it unnecessary to keep track of the "ill-definedness" of $\varrho_\Phi(R)$ as an operator on the string Hilbert space. It is analytic almost everywhere, and its behaviour at the singular points is well controlled. Every calculation may be performed *as if* $\mathrm{ad}\varrho_\Phi(R)$ where an inner derivation.

4. Transformation of Operators

The form (2.7) of the generator of a deformation $\varrho_\Phi(R)$ opens for calculations of commutators as integrals of correlation functions over the complex plane, instead of making direct use of the mode expansions, much in the same spirit as one uses contour integrals in a holomorphic theory. The transformation of a local field $\Psi(z,\bar{z})$ is

$$\delta_\Phi \Psi(z,\bar{z}) = \Big[-\frac{1}{\pi} \int_{|w|<|z|} d^2 w\, \Phi(w,\bar{w})\ ,\ \Psi(z,\bar{z}) \Big]\ . \tag{4.1}$$

Using the property (3.4) this is rewritten as

$$\begin{aligned} \delta_\Phi \Psi(z,\bar{z}) &= \frac{1}{\pi} \Big\{ \int_{|w|>|z|} d^2 w\, \Phi(w,\bar{w}) \Psi(z,\bar{z}) + \int_{|w|<|z|} d^2 w\, \Psi(z,\bar{z}) \Phi(w,\bar{w}) \Big\} \\ &= \frac{1}{\pi} \int_{\mathbb{C}} d^2 w\, \mathscr{R} \left[\Phi(w,\bar{w}) \Psi(z,\bar{z}) \right]\ , \end{aligned} \tag{4.2}$$

$\mathscr{R}$ denoting radial ordering. This is a desirable expression, since radial ordering is exactly what is needed in order for the mode expansions of normal ordering terms to converge. It also means,

in the light of (3.4), that regular terms in the operator product of (4.2) do not contribute to the commutator.

The drawback of expressions like this, as compared to contour integrals in a holomorphic theory, is that there is no analogy to deformation of integration contours. It is not allowed to expand $\Phi(w,\bar{w})$ in a Taylor series around $w=z$, since such an expansion only converges inside the circle $|w-z|<|z|$.

The first thing to check is that (4.2) gives the correct result for $\delta_\Phi T(z,\bar{z})$ (the calculation for $\bar{T}$ is analogous). We thus have

$$\begin{aligned}\delta_\Phi T(z,\bar{z}) &= \frac{1}{\pi}\int_{\mathbb{C}} d^2w\, \mathscr{R}\left[T(z)\Phi(w,\bar{w})\right]\\ &= \frac{1}{\pi}\int_{\mathbb{C}} d^2w\, \left\{\frac{1}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial\Phi(w,\bar{w}) + \text{regular}\right\}\ ,\end{aligned} \tag{4.3}$$

where the explicit functions of $z-w$ are defined through their convergent series expansions in the regions $|w|<|z|$ and $|w|>|z|$. Splitting the integration region and expanding the series gives for each term in the expansion (2.2) of Φ an integral of the type

$$\begin{aligned}J(\alpha,m;n;z) &= \frac{1}{\pi}\int_{\mathbb{C}} d^2w\, |w|^{2\alpha}w^m(z-w)^n\\ &= |z|^{2(\alpha+1)}z^{m+n}\frac{1}{\pi}\int_{\mathbb{C}} d^2w\, |w|^{2\alpha}w^m(1-w)^n = |z|^{2(\alpha+1)}z^{m+n}J_0(\alpha,m;n)\ .\end{aligned} \tag{4.4}$$

We calculate $J_0(\alpha,m;n)$ as

$$\begin{aligned}J_0(\alpha,m;n) =& \frac{1}{\pi}\int_{|w|<1} d^2w\, \sum_{k=0}^{\infty}(-1)^k\binom{n}{k}|w|^{2\alpha}w^{m+k}\\ &+\frac{1}{\pi}\int_{|w|>1} d^2w\, \sum_{k=0}^{\infty}(-1)^{n+k}\binom{n}{k}|w|^{2\alpha}w^{m+n-k}\ .\end{aligned} \tag{4.5}$$

The first integral contributes only when $m\leq 0$, its value is then $\frac{(-1)^m}{1+\alpha}\binom{n}{-m}$; the second one when $m+n\geq 0$ with the value $-\frac{(-1)^m}{1+\alpha}\binom{n}{n+m}$. It is obvious that the two terms cancel when $n\geq 0$, i.e. when the integrand only has singularities at $w=0$ or ∞. When $n=-N$, $N=1,2,\ldots$, the two terms combine (for any $\alpha\neq -1$ and m) to give

$$J_0(\alpha,m;-N) = \frac{1}{1+\alpha}\frac{(1-m)_{N-1}}{(N-1)!} \tag{4.6}$$

(see Appendix for notation), which in particular means that if we let $f(z,\bar{z})=|z|^{2\alpha}z^m$, we obtain

$$\frac{1}{\pi}\int_{\mathbb{C}} d^2w\, \left\{\frac{1}{(1-w)^2}f(w,\bar{w}) + \frac{1}{1-w}\partial f(w,\bar{w})\right\} = \frac{1-m}{1+\alpha} + \frac{\alpha+m}{1+\alpha} = 1\ , \tag{4.7}$$

and, in view of (4.4),

$$\frac{1}{\pi}\int_{\mathbb{C}} d^2w\, \left\{\frac{1}{(z-w)^2}f(w,\bar{w}) + \frac{1}{z-w}\partial f(w,\bar{w})\right\} = |z|^{2(\alpha+1)}z^{m-2} = \frac{\bar{z}}{z}f(z,\bar{z})\ , \tag{4.8}$$

thus verifying equation (2.1) for the variation of the stress tensor.

The next natural thing to examine is how physical vertex operators deform. Equation (4.2) contains this information, although in rather implicit form. Any explicit formula will depend on the detailed behaviour of the operator product $\Phi(w,\bar{w})\Psi(z,\bar{z})$. If we also take Ψ to be a physical vertex operator, $\delta_\Phi\Psi(z,\bar{z})$ is the operator containing the four-string amplitudes with any two states besides Φ and Ψ. Up to a constant:

$$|z|^{-2}A_4(\mathscr{V},\mathscr{V}',\Phi,\Psi)=\frac{1}{\pi}\int\limits_{\mathbb{C}}d^2w\ <\mathscr{V}|\mathscr{R}\left[\Phi(w,\bar{w})\Psi(z,\bar{z})\right]|\mathscr{V}'>=<\mathscr{V}|\delta_\Phi\Psi(z,\bar{z})|\mathscr{V}'>\quad . \tag{4.9}$$

Since no Taylor expansion around $w=z$ is allowed, we have to evaluate the integral for each mode of Φ. We perform this calculation for the deformation of a tachyon by another tachyon as an example. Any other case goes along the same lines, and no technical difficulties are left out by this example. In order to give a more general formula, one would have to resort to the DDF construction [13] of physical vertex operators for states of arbitrary m^2.

Let the two tachyon vertices be $\Phi(z,\bar{z})=\exp(ik\cdot X(z,\bar{z}))$ and $\Psi(z,\bar{z})=\exp(ik'\cdot X(z,\bar{z}))$, with $k^2=k'^2=8$. The operator product is

$$\Phi(w,\bar{w})\Psi(z,\bar{z})=|z-w|^{\frac{k\cdot k'}{2}}e^{ik\cdot X(w,\bar{w})+ik'\cdot X(z,\bar{z})}\ . \tag{4.10}$$

The integrals to be evaluated are of the type

$$\begin{aligned}I(\alpha,m;\beta,n;z)&=\frac{1}{\pi}\int\limits_{\mathbb{C}}d^2w\,|w|^{2\alpha}w^m|z-w|^{2\beta}(z-w)^n\\&=|z|^{2(1+\alpha+\beta)}z^{m+n}\frac{1}{\pi}\int\limits_{\mathbb{C}}d^2w\,|w|^{2\alpha}w^m|1-w|^{2\beta}(1-w)^n\\&=|z|^{2(1+\alpha+\beta)}z^{m+n}I_0(\alpha,m;\beta,n)\ .\end{aligned} \tag{4.11}$$

In our specific example we have $n=0$, but a generic operator product will involve the general form.

The value of $I_0(\alpha,m;\beta,n)$ is known [14]. It is

$$I_0(\alpha,m;\beta,n)=\frac{\Gamma(1+\alpha+m)\Gamma(1+\beta+n)\Gamma(-1-\alpha-\beta)}{\Gamma(-\alpha)\Gamma(-\beta)\Gamma(2+\alpha+\beta+m+n)}\ . \tag{4.12}$$

In ref. [14], this integral was calculated by the standard method for evaluating tree-level string amplitudes. In principle, $I_0(\alpha,m;\beta,n)$ may also be calculated using series expansions for the functions of $z-w$ and splitting the integration region in $|w|<|z|$ and $|w|>|z|$. Then the calculation reads $I_0=I_<+I_>$, and

$$I_<(\alpha,m;\beta,n)=\frac{1}{\pi}\int\limits_{|w|<1}d^2w\,|w|^{2\alpha}w^m\sum_{k,l=0}^{\infty}(-1)^{k+l}\binom{\beta+n}{k}\binom{\beta}{l}w^k\bar{w}^l\ . \tag{4.13}$$

The angular integration restricts the sum to those terms which have $m+k=l$. One has to distinguish the cases $m\geq 0$ and $m\leq 0$. The remaining sum is collected in a hypergeometric

function (see Appendix). The calculation of $I_>$ is analogous, and the complete result is

$$
\begin{aligned}
I_0(\alpha,m;\beta,n) &= \begin{cases} \frac{\Gamma(-\beta+m)}{\Gamma(-\beta)\Gamma(1+m)} \int_0^1 dx\, x^{\alpha+m}\, {}_2F_1(-\beta-n,-\beta+m;1+m;x) & (m\geq 0) \\ \frac{\Gamma(-\beta-m-n)}{\Gamma(-\beta-n)\Gamma(1+m)} \int_0^1 dx\, x^{\alpha}\, {}_2F_1(-\beta,-\beta-m-n;1-m;x) & (m\leq 0) \end{cases} \\
&+ \begin{cases} (-1)^n \frac{\Gamma(-\beta+m)}{\Gamma(-\beta-n)\Gamma(1+m+n)} \int_0^1 dx\, x^{-2-\alpha-\beta}\, {}_2F_1(-\beta,-\beta+m;1+m+n;x) \\ \qquad\qquad (m+n\geq 0) \\ (-1)^n \frac{\Gamma(-\beta-m-n)}{\Gamma(-\beta)\Gamma(1-m-n)} \int_0^1 dx\, x^{-2-\alpha-\beta-m-n}\, {}_2F_1(-\beta-n,-\beta-m-n;1-m-n;x) \\ \qquad\qquad (m+n\leq 0) \end{cases} \\
&= \begin{cases} \frac{1}{1+\alpha+m} \frac{\Gamma(-\beta+m)}{\Gamma(-\beta)\Gamma(1+m)}\, {}_3F_2(-\beta-n,-\beta+m,1+\alpha+m;1+m,2+\alpha+m;1) & (m\geq 0) \\ \frac{1}{1+\alpha} \frac{\Gamma(-\beta-m-n)}{\Gamma(-\beta-n)\Gamma(1-m)}\, {}_3F_2(-\beta,-\beta-m-n,1+\alpha;1-m,2+\alpha;1) & (m\leq 0) \end{cases} \\
&+ \begin{cases} \frac{(-1)^{n+1}}{1+\alpha+\beta} \frac{\Gamma(-\beta+m)}{\Gamma(-\beta-n)\Gamma(1+m+n)}\, {}_3F_2(-\beta,-\beta+m,-1-\alpha-\beta;1+m+n,-\alpha-\beta;1) \\ \qquad\qquad (m+n\geq 0) \\ \frac{(-1)^{n+1}}{1+\alpha+\beta+m+n} \frac{\Gamma(-\beta-m-n)}{\Gamma(-\beta)\Gamma(1-m-n)} \\ \qquad \times\, {}_3F_2(-\beta-n,-\beta-m-n,-1-\alpha-\beta-m-n;1-m-n,-\alpha-\beta-m-n;1) \\ \qquad\qquad (m+n\leq 0)\ . \end{cases}
\end{aligned}
\tag{4.14}
$$

The regularization is now hidden in the transition from hypergeometric *series* to hypergeometric *function*, the latter being an analytic continuation of the former, identical to ζ-function regularization of the sum.

While the value of ${}_2F_1$ at 1 is given by the Gauss formula,

$$
{}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ , \tag{4.15}
$$

the general value of ${}_3F_2$ at 1 can not be expressed in terms of simpler functions, such as the Γ-function, apart from some special cases [15]. The integrals we have at hand do not seem to belong to these. On the other hand, an alternative calculation [14] has already shown that the sum $I_< + I_>$ simplifies to (4.12). The difficulty with the calculation we are trying to perform is that the pole structure at $w=1$ is treated in an asymmetric fashion. The boundary line between the two integration regions goes through $w=1$, so that each of $I_<$ and $I_>$ aquires poles from integration over a small semicircle around $w=1$. Many of these poles cancel between the two integrals. By examining the behaviour of the integrands in (4.14) at $w=1$, using the technique of the Appendix, we see that there are poles for half-integer values of β, but when the contributions from $I_<$ and $I_>$ are added, only the integer ones survive. They correspond to the physical resonances in $\delta_\Phi\Psi$. The residues of the poles can be expressed as finite sums, and we have checked our calculation by comparing these with the pole structure of (4.12).

We would like to remark on a curious and important fact. The integral $J_0(\alpha,m;n)$ already calculated seems to be identical to $I_0(\alpha,m;0,n)$, so one would expect the former to be obtained from the latter as $\lim_{\beta\to 0} I_0(\alpha,m;\beta,n)$, as an analytic function of α. For $n\geq 0$ this gives identically zero, but for $n=-N$, $N=1,2,\ldots$ there is a removable singularity due to the presence of poles

both in the nominator and the denominator. The unique limit is

$$\lim_{\beta\to 0} I_0(\alpha,m;\beta,-N)=\frac{1}{1+\alpha}\frac{(-\alpha-m)_{N-1}}{(N-1)!}\ , \tag{4.16}$$

which is obviously not in agreement with (4.6). The difference can be interpreted as follows. The procedure of splitting the integration region may be viewed as omitting the annulus

$$D_\varepsilon=\{w\in\mathbb{C}\mid 1-\varepsilon<|w|<1+\varepsilon\} \tag{4.17}$$

from the integration region, and letting $\varepsilon\to 0$. Since D_ε contains the point $w=1$ where we have potential singularities, it is not obvious that the integral over D_ε will vanish in the limit $\varepsilon\to 0$. We can exemplify this with the integrand $(1-w)^{-2}$, where it is not too difficult to evaluate the integral explicitly in the limit $\varepsilon\to 0$ (D_ε can effectively be replaced by $\{w\,|\,1-\varepsilon<\mathrm{Re}\,w<1+\varepsilon\}$ and by symmetry this can in turn be reduced to $\{w\,|\,1-\varepsilon<\mathrm{Re}\,w<1+\varepsilon\ ,\ |\mathrm{Im}\,w|>\varepsilon\}$). We obtain

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{D_\varepsilon}\frac{d^2w}{(1-w)^2}=-1\ , \tag{4.18}$$

which exactly matches the difference between $I_0(0,0;0,-2)$ and $J_0(0,0;-2)$, thus adding support to our interpretation. The choice of evaluating J_0 as we did, i.e., of effectively integrating over $\mathbb{C}\setminus D_\varepsilon$, has a simple physical motivation — it corresponds to radial point-splitting of the operators in the commutator. In the generic analytic case it is seen to be of no importance, but for the deformation of $T(z)$ it was crucial: while momenta are good variables for analytic continuation, the conformal dimension of $T(z)$ is not. We would like to emphasize clearly that the physically correct result is always produced by splitting the integral in $I_<$ and $I_>$, as implied by the series expansion of the normal ordering terms. Without giving a strict argument, discrepancies between the analytic expression and the result of a point-split series expansion should occur exactly when the difference between the inner and outer expansions has a meaning as a non-vanishing distribution on the circle $|w|=1$.

After this excursion, we return to the example with tachyonic deformation of a tachyon. We replace the scalar products of momenta by Mandelstam variables according to

$$\begin{aligned} s&=-(k+p)^2\ ,\\ t&=-(k+k')^2\ ,\\ u&=-(k'+p)^2\ , \end{aligned} \tag{4.19}$$

and define $p'=-k-k'-p$. Then $s+t+u+p^2+p'^2+16=0$ (note that p and thus s and u are operator valued). Inserting the integral (4.12) in (4.2) using the operator product (4.10) gives the result

$$\begin{aligned} \delta_\Phi\Psi(z,\bar z)=&\sum_{m,n}\frac{\Gamma(-\frac{s}{8}-1-m-\frac{p^2}{8})\Gamma(-\frac{t}{8}-1)\Gamma(-\frac{u}{8}+1+n-\frac{p'^2}{8})}{\Gamma(\frac{s}{8}+2+n+\frac{p^2}{8})\Gamma(\frac{t}{8}+2)\Gamma(\frac{u}{8}-m+\frac{p'^2}{8})}\\ &\qquad\times\ :\Phi_{mn}\Psi(z,\bar z):\ |z|^{\frac{u}{4}+\frac{p'^2}{4}}z^{-1-m}\bar z^{-1-n}\\ =&\sum_{M,N,m,n}\frac{\Gamma(-\frac{s}{8}-1-m-\frac{p^2}{8})\Gamma(-\frac{t}{8}-1)\Gamma(-\frac{u}{8}+1+n-\frac{p'^2}{8})}{\Gamma(\frac{s}{8}+2+n+\frac{p^2}{8})\Gamma(\frac{t}{8}+2)\Gamma(\frac{u}{8}-m+\frac{p'^2}{8})}\\ &\qquad\times\ :\Phi_{mn}\Psi_{M-m,N-n}:\ |z|^{-4-\frac{p^2}{4}+\frac{p'^2}{4}}z^{-1-M}\bar z^{-1-N}\ . \end{aligned} \tag{4.20}$$

This operator contains as its $|z|^{-2}$-component all four-string amplitudes of two tachyons and any other two physical states. This term has $M=N=-\Delta-2$, where $\Delta=\frac{1}{8}(p^2-p'^2)$ is the shift in excitation number from the in-state to the out-state. It is easy to show, using $\Gamma(x)\Gamma(1-x)=\pi/\sin(\pi x)$, that it is symmetric under the exchange of Φ and Ψ, i.e., the quotient of Γ-functions in eq. (4.20) is invariant under $m\leftrightarrow-\Delta-2-m$, $n\leftrightarrow-\Delta-2-n$, $s\leftrightarrow u$. Especially it contains the well-known four-tachyon amplitude at $M=N=-2$, $m=n=-1$:

$$A_{\text{4-tachyon}}(s,t,u)\sim\frac{\Gamma(-\frac{s}{8}-1)\Gamma(-\frac{t}{8}-1)\Gamma(-\frac{u}{8}-1)}{\Gamma(\frac{s}{8}+2)\Gamma(\frac{t}{8}+2)\Gamma(\frac{u}{8}+2)}\ . \tag{4.21}$$

A more general class of operators that naturally enter string amplitudes are the multi-local operators of the type

$$\mathscr{W}^{(N)}(z_1,\bar z_1;\ldots;z_N,\bar z_N)=\mathscr{R}\,[\,\prod_{i=1}^{N}\mathscr{V}_i(z_i,\bar z_i)\,]\ . \tag{4.22}$$

Consider the product (4.22) with $|z_1|>|z_2|>\ldots>|z_N|$. Then, using (4.2) and (3.4), and letting each local field transform as

$$\begin{aligned}
\delta_\Phi\mathscr{W}^{(N)}(z_1,\bar z_1;\ldots;z_N,\bar z_N)&=\sum_{i=1}^{N}\mathscr{V}_1(z_1,\bar z_1)\ldots\delta_\Phi(|z_i|)\mathscr{V}_i(z_i,\bar z_i)\ldots\mathscr{V}_N(z_N,\bar z_N)\\
&=\frac{1}{\pi}\int\limits_{|w|>|z_1|}d^2w\,\Phi(w,\bar w)\mathscr{V}_1(z_1,\bar z_1)\ldots\mathscr{V}_N(z_N,\bar z_N)\\
&+\frac{1}{\pi}\sum_{i=1}^{N-1}\mathscr{V}_1(z_1,\bar z_1)\ldots\mathscr{V}_i(z_i,\bar z_i)\Big(\int\limits_{|w|<|z_i|}+\int\limits_{|w|>|z_{i+1}|}\Big)d^2w\,\Phi(w,\bar w)\mathscr{V}_{i+1}(z_{i+1},\bar z_{i+1})\ldots\mathscr{V}_N(z_N,\bar z_N)\\
&+\frac{1}{\pi}\int\limits_{|w|<|z_N|}d^2w\,\mathscr{V}_1(z_1,\bar z_1)\ldots\mathscr{V}_N(z_N,\bar z_N)\Phi(w,\bar w)\\
&=\frac{1}{\pi}\int\limits_{|w|>|z_1|}d^2w\,\Phi(w,\bar w)\mathscr{V}_1(z_1,\bar z_1)\ldots\mathscr{V}_N(z_N,\bar z_N)\\
&+\frac{1}{\pi}\sum_{i=1}^{N-1}\int\limits_{|z_{i+1}|<|w|<|z_i|}d^2w\mathscr{V}_1(z_1,\bar z_1)\ldots\mathscr{V}_i(z_i,\bar z_i)\Phi(w,\bar w)\mathscr{V}_{i+1}(z_{i+1},\bar z_{i+1})\ldots\mathscr{V}_N(z_N,\bar z_N)\\
&+\frac{1}{\pi}\int\limits_{|w|<|z_N|}d^2w\,\mathscr{V}_1(z_1,\bar z_1)\ldots\mathscr{V}_N(z_N,\bar z_N)\Phi(w,\bar w)\\
&=\frac{1}{\pi}\int\limits_{\mathbb{C}}d^2w\,\mathscr{R}\,[\,\Phi(w,\bar w)\prod_{i=1}^{N}\mathscr{V}_i(z_i,\bar z_i)\,]\ .
\end{aligned} \tag{4.23}$$

This means that the N-string amplitudes pick up variations containing the $(N+1)$-string amplitudes, under the condition that the vacua at $R=0$ and $R=\infty$ do not transform (which actually

follows from our regularization). It is also obvious that an argument for background invariance [16] has to involve transformations on the multi-string Fock space — no transformation of the external states at $R=0,\infty$ contracting (4.22) can compensate for the transformations (4.23). As amplitudes are constructed from operators at different radii, the possibility of treating deformations as "almost inner automorphisms" is not relevant for invariance of expectation values of expressions like (4.22) unless all operators and states reside at equal radii. Equation (4.23) should provide the natural connection from our canonical first-quantized formalism to transformations on the string field theory Fock space.

It is important to note that eq. (4.23) relies directly on the regularization used. It is of course the most natural thing to write down, but in other regularization schemes it will not be consistent with the transformations of local fields.

5. Commutators of Deformations

We have treated the deformations as though they were inner derivations of the conformal field theory. However, the resonances occurring for deformations of physical vertex operators imply that this is not really true — the resonances are exactly what makes the transformations non-trivial, and the deformed theory inequivalent to the undeformed one, simply in the sense that they are not related by an automorphism. The deformations as we have defined them, constitute the unique prescription (modulo inner derivations) for parallel transport of states and operators that respect conformal invariance. The resonances are essential, and necessary to make the deformations non-trivial.

The corresponding connection has several interesting local properties. An important consequence of our regularization prescription, namely of the result (3.3), is that the connection is compatible with the Zamolodchikov metric:

$$\delta_\Phi \mathbf{1} = \frac{1}{\pi}\int_{\mathbb{C}} d^2z\,\Phi(z,\bar z) = 0\ . \tag{5.1}$$

The curvature of the connection is contained in the commutators of any two deformations. The necessary calculations have been performed in the previous chapter and we find that the curvature vanishes:

$$\begin{aligned}&[\,\delta_\Phi,\delta_\Psi\,]\mathscr{W}^{(N)}(z_1,\bar z_1;\ldots;z_N,\bar z_N) =\\ &= \delta_\Phi\frac{1}{\pi}\int_{\mathbb{C}} d^2w\,\mathscr{R}\,[\,\Psi(w,\bar w)\prod_{i=1}^{N}\mathscr{V}_i(z_i,\bar z_i)\,] - \delta_\Psi\frac{1}{\pi}\int_{\mathbb{C}} d^2w\,\mathscr{R}\,[\,\Phi(w,\bar w)\prod_{i=1}^{N}\mathscr{V}_i(z_i,\bar z_i)\,]\\ &= \frac{1}{\pi^2}\int_{\mathbb{C}} d^2w\int_{\mathbb{C}} d^2z\,\mathscr{R}\,[\,(\Phi(w,\bar w)\Psi(z,\bar z)-\Psi(w,\bar w)\Phi(z,\bar z))\prod_{i=1}^{N}\mathscr{V}_i(z_i,\bar z_i)\,] = 0\ .\end{aligned} \tag{5.2}$$

It is important to note that if we would have computed the commutator between two deformations in a strictly first-quantized formalism then we would have been led to the commutator

between the two generators of the form (2.7). This commutator is however non-zero:

$$
\begin{aligned}
[\,\varrho_\Phi(R),\varrho_\Psi(R)\,] &= \frac{1}{\pi^2}\int\limits_{\substack{|z|<R\\|w|<R}} d^2z d^2w\,[\,\Phi(w,\bar w),\Psi(z,\bar z)\,]\\
&= \frac{1}{\pi^2}\int\limits_{\substack{|z|>R\\|w|<R}} d^2z d^2w\,\Psi(z,\bar z)\Phi(w,\bar w) - \frac{1}{\pi^2}\int\limits_{\substack{|z|<R\\|w|>R}} d^2z d^2w\,\Phi(w,\bar w)\Psi(z,\bar z)\\
&= \frac{1}{\pi^2}\Big(\int\limits_{\substack{|z|>R\\|w|<R}} - \int\limits_{\substack{|z|<R\\|w|>R}}\Big) d^2z d^2w\,\mathscr{R}\,[\,\Phi(w,\bar w)\Psi(z,\bar z)\,]\\
&= \frac{1}{\pi^2}\Big(\int\limits_{\substack{|z|\in\mathbb{C}\\|w|<R}} - \int\limits_{\substack{|z|<R\\|w|\in\mathbb{C}}}\Big) d^2z d^2w\,\mathscr{R}\,[\,\Phi(w,\bar w)\Psi(z,\bar z)\,]\\
&= \varrho_{\delta_\Phi\Psi-\delta_\Psi\Phi}(R)\ .
\end{aligned}
\tag{5.3}
$$

From the fact that the amplitude (4.9) is independent of z for physical $|\mathscr{V}\rangle$, $|\mathscr{V}'\rangle$, it follows that all z dependent components of $\delta_\Phi\Psi(z,\bar z)$ are trivial operators, i.e., vanish on the physical Hilbert space (this space being considered as consisting of equivalence classes of physical states modulo trivial states). Using the symmetry of physical amplitudes, one concludes that the entire operator $\delta_\Phi\Psi(z,\bar z)-\delta_\Psi\Phi(z,\bar z)$ is trivial but nevertheless non-zero.

The curvature tensor (5.2) thus vanishes on the full state space, which of course means that its pullback to the physical state space vanishes. If one instead considers the pullback of the connection δ_Φ to the physical state space, one finds that it carries non-zero curvature, which is consistent with what is known concerning the Zamolodchikov metric.

Further examination in the second-quantized formalism of the local properties of the moduli space of conformal field theories, such as the finite parallel transport generated by the connection and the construction of a connection for the tangent bundle of the moduli space, will be deferred to [2].

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Appendix: Hypergeometric Functions

The hypergeometric series ${}_AF_B(a_1,\ldots,a_A;b_1\ldots,b_B;z)$ [15] is defined as

$$
\begin{aligned}
{}_AF_B(a_1,\ldots,a_A;b_1\ldots,b_B;z)&=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_A)_n}{n!\,(b_1)_n\ldots(b_B)_n}\,z^n\\
&=\frac{\Gamma(b_1)\ldots\Gamma(b_B)}{\Gamma(a_1)\ldots\Gamma(a_A)}\sum_{n=0}^{\infty}\frac{\Gamma(a_1+n)\ldots\Gamma(a_A+n)}{n!\,\Gamma(b_1+n)\ldots\Gamma(b_B+n)}\,z^n\ ,
\end{aligned}
\tag{A.1}
$$

where $(x)_k$ is the Pochhammer symbol $(x)_k=x(x+1)\ldots(x+k-1)$. The relevant hypergeometric functions in the present application are those with $A=B+1$. The hypergeometric functions constitute the analytic continuation of the hypergeometric series (A.1) outside of the convergence region of the series. They are analytic for $z\in\mathbb{C}$, except for the possibility of poles at $z=0$ and a cut from $z=1$ to $z=\infty$ (depending on the values of the parameters). The asymptotic properties of the series (A.1) can be investigated using the asymptotic behaviour of the Γ-function,

$$
\log\Gamma(n)=(n-\frac{1}{2})\log n-n+\log\sqrt{2\pi}+\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}n^{1-2k}\ ,
\tag{A.2}
$$

where B_{2k} are Bernoulli numbers. This asymptotic expansion applied to the quotient of two Γ-functions gives an expansion

$$
\frac{\Gamma(n+a)}{\Gamma(n+b)}=n^{a-b}\sum_{k=0}^{\infty}c_k(a,b)n^{-k}\ .
\tag{A.3}
$$

The class of hypergeometric series having interesting convergence properties is therefore the case $A=B+1$. The convergence criterion at $z=1$ is $\sum a-\sum b<0$. The analytic continuation can be seen as ζ-function regularization of the series. The pole of the ζ-function $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$ at $s=1$ with residue 1 gives poles at $\sum a-\sum b=0,1,2\ldots$ for the hypergeometric funtion at $z=1$, i.e., exactly when the sum $\sum\frac{1}{n}$ occurs from (A.3).

The value of ${}_2F_1$ at $z=1$ is given by the Gauss formula

$$
{}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ ,
\tag{A.4}
$$

but the corresponding general value of ${}_{A+1}F_A$ at $z=1$ can not be expressed in terms of simpler functions for generic values of the a and b parameters. During the evaluation of some integrals in section 4, we wanted to find the value of

$$
\int_0^1 dx\,x^p\,{}_2F_1(a,b;c;x)=\frac{1}{p+1}\,{}_3F_2(a,b,p+1;c,p+2;1)\ .
\tag{A.5}
$$

We can derive an expression for the pole structure of (A.5) as follows. The lower integration limit is simple, one picks up poles (using the regularization (3.2)) when $p=-N$, $N=1,2,\ldots$, with the residues $\frac{(a)_N(b)_N}{N!(c)_N}$. For the upper limit, we may use a “reflection formula” for the hypergeometric

function. With the same argument as above for the value of the function at $z=1$, we conclude that poles only occur when $c-a-b=-N$, $N=1,2,\ldots$, and [15]

$$
\begin{aligned}
{}_2F_1(a,b;a+b-N;x) &= \frac{\Gamma(N)\Gamma(a+b-N)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{N-1}\frac{(a-N)_k(b-N)_k}{k!\,(1-N)_k}(1-x)^{k-N}\\
&\quad-(-1)^N\frac{\Gamma(a+b-N)}{\Gamma(a-N)\Gamma(b-N)}\\
&\qquad\times\sum_{k=0}^{\infty}\big[\log(1-x)-\psi(1+k)-\psi(1+N+k)+\psi(a+k)+\psi(b+k)\big](1-x)^k\ ,
\end{aligned}
\tag{A.6}
$$

where $\psi(z)$ is the digamma function $\psi(z)=\frac{d}{dz}\log\Gamma(z)$ (the cut from $z=1$ to $z=\infty$ of the hypergeometric function becomes logarithmic for these values of the parameters). The only part of (A.6) that can develop poles upon integration weighted with a function that is regular at $x=1$ is the finite sum. We expand the function x^p in a power series around $x=1$ and identify the coefficient of the $(1-x)^{-1}$-term, that gives the residue of the pole. The residue is, after a short calculation,

$$
\mathrm{Res}_{c=a+b-N}\int_0^1 dx\,x^p\,{}_2F_1(a,b;c;x)=\frac{\Gamma(a+b-N)(-p)_{N-1}}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{N-1}\frac{(a-N)_k(b-N)_k}{k!\,(2+p-N)_k}\ . \tag{A.7}
$$

In terms of the integrals $I_<$ and $I_>$ of section 4, we have as an example

$$
\mathrm{Res}_{\beta=-\frac{1+N+n}{2}}I_<(\alpha,m\geq 0;\beta,n)=\frac{(-\alpha-m)_{N-1}}{2\Gamma(\frac{1+N+n}{2})\Gamma(\frac{1+N-n}{2})}\sum_{k=0}^{N-1}\frac{\big(\frac{1-N-n}{2}\big)_k\big(\frac{1-N+n}{2}+m\big)_k}{k!\,(2+\alpha+m-N)_k}\ . \tag{A.8}
$$

We note that the factors in front of the sum makes the residue vanish for $N+n$ odd, $|n|>N$. In fact, the two terms in equation (4.14) cancel for $N+n$ even, so the remaining poles lie at $n=-N+1,-N+3,\ldots,N-3,N-1$, i.e., only for negative integer β. We are convinced that the finite sums for the residues can be simplified, but since the result for the integral already is known, we have contended ourselves to check the equality of (A.8) with the residues of the known formula (4.12) using *Mathematica*. We remind that although (4.12) gives the correct analytic expression for the integral, the limit of this analytic function does not give the right result in some (a discrete set of) cases. This is due to radial point-splitting of operator products, as introduced in section 4. This discrepancy occurred for the deformation of the stress tensor, whose conformal dimension is not a good variable for analytic continuation. The equality of equations (4.14) and (4.12) does not seem to be known in the mathematics literature.

REFERENCES

[1] B. Zwiebach, *Nucl.Phys.* **B390** (1993) 33, hep-th/9206084;
B. Zwiebach, hep-th/9305026.

[2] A. von Gussich and P. Sundell, in preparation.

[3] K. Ranganathan, *Nucl.Phys.* **B408** (1993) 180, hep-th/9210090.

[4] K. Ranganathan, H. Sonoda and B. Zwiebach, *Nucl.Phys.* **B414** (1994) 405, hep-th/9304053.

[5] G. Pelts, *Phys.Rev.* **D51** (1995) 671, hep-th/9406022.

[6] T. Kugo, B. Zwiebach, *Prog.Theor.Phys.* **87** (1992) 801-860.

[7] S. Hawking, *Commun.Math.Phys.* **55** (1977) 133.

[8] M. Evans and B. Ovrut, *Phys.Rev.* **D39** (1989) 3016.

[9] M. Campbell, P. Nelson and E. Wong, *Int.J.Mod.Phys.* **A6** (1991) 4909.

[10] M. Evans, I. Giannakis and D.V. Nanopoulos, *Phys.Rev.* **D50** (1994) 4022, hep-th/9401075.

[11] G.J. Zuckerman, talk presented at the $1^{\underline{st}}$ Feza Gürsey Memorial Conference on Strings and Symmetries, Istanbul, Turkey, 1994.

[12] M. Evans and B. Ovrut, *Phys.Rev.* **D41** (1990) 3149.

[13] E. Del Giudice, P. Di Vecchia and S. Fubini, *Ann.Phys.* **70** (1972) 378.

[14] D. Gross, J. Harvey, E. Martinec and R. Rohm, *Nucl.Phys.* **B267** (1986) 75.

[15] H. Exton, "Handbook of Hypergeometric Integrals", Ellis Horwood, Chichester, 1978;
M. Abramowitz and I.A. Stegun (eds.), "Handbook of Mathematical Functions", National Bureau of Standards, Washington, 1972.

[16] A. Sen and B. Zwiebach, *Nucl.Phys.* **B414** (1994) 649, hep-th/9307088;
A. Sen and B. Zwiebach, *Nucl.Phys.* **B423** (1994) 580, hep-th/9311009.